L11 – Week 6
Min-max Optimization: Local Nash and Last iterate convergence

CS 295 Optimization for Machine Learning

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Min-max in bilinear

- Previously we motivated the Last iterate convergence.
- We show that Gradient Descent Ascent (GDA) diverges even for $x^T Ay$.

Intuition: Given the bilinear problem below let's run the continuous GDA.

$$\min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} x^T Ay.$$

Consider continuous GDA that is the system of odes:

Recall GDA:

$$x_{t+1} = x_t - \eta \nabla_x f(x_t, y_t),$$
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Lemma (Cycles). *It holds that* $||x||_2^2 + ||y||_2^2$ *is constant w.r.t* t.

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Proof. It suffices to prove

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Observe that

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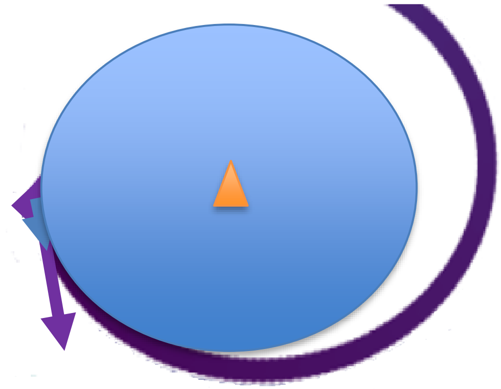
Hence

$$\frac{d}{dt}\{\|x\|_2^2 + \|y\|_2^2\} = -2\eta x^T Ay + 2\eta x^T Ay = 0.$$

Min-max in bilinear



- Question: Can we fix this behavior? We can use "optimism" (negative momentum).

$$x_{t+1} = x_t - \eta \cdot \nabla_x f(x_t, y_t) + \boldsymbol{\eta/2 \cdot \nabla_x f(x_{t-1}, y_{t-1})}$$

$$y_{t+1} = y_t + \eta \cdot \nabla_y f(x_t, y_t) - \boldsymbol{\eta/2 \cdot \nabla_y f(x_{t-1}, y_{t-1})}$$

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Min-max in bilinear (OGDA)

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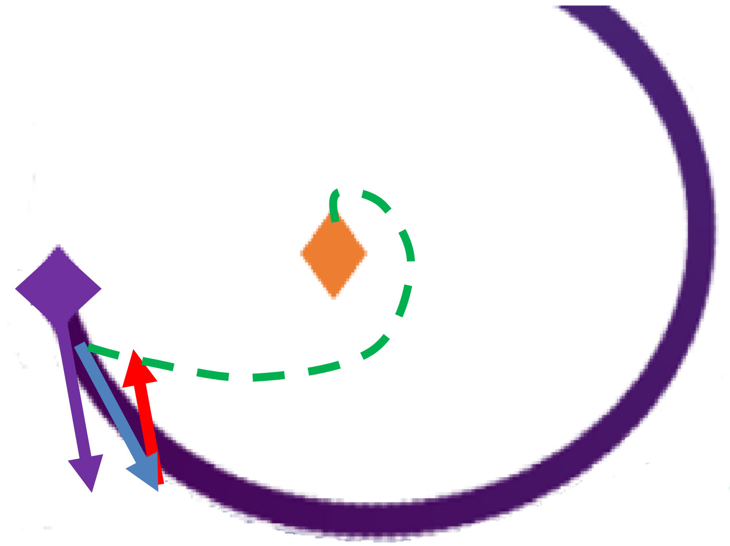

Min-max in bilinear (OGDA)

Theorem (Convergence). *Consider the bilinear game $x^T Ay$ where A is full rank. Optimistic GDA converges pointwise and reaches an ϵ neighborhood in*

$$T := \Theta\left(\frac{\lambda_{\max}(AA^T)}{\lambda_{\min}(AA^T)} \log \frac{1}{\epsilon}\right)$$

choosing learning rate $\eta = \frac{1}{4\sqrt{\lambda_{\max}(AA^T)}}$.

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The idea behind the proof is to analyze the following dynamical system

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \left(I - \begin{pmatrix} 0 & 2\eta A \\ -2\eta A^T & 0 \end{pmatrix}\right)\begin{pmatrix} x_t \\ y_t \end{pmatrix} + \eta \begin{pmatrix} 0 & 2\eta A \\ -2\eta A^T & 0 \end{pmatrix}\begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix}$$

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Let's make it linear system!

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Consider the linear dynamical system

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Observe that

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ x_t \\ y_t \end{pmatrix} = \begin{pmatrix} I & -2\eta A & 0 & \eta A \\ 2\eta A^T & I & -\eta A^T & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ x_{t-1} \\ y_{t-1} \end{pmatrix}$$

Lemma (Eigenvalues). *The matrix above has eigenvalues that are less than one for the appropriate choice of η.*

Min-max in bilinear (constrained)

Consider the problem

$$\min_{x \in \Delta_n} \max_{y \in \Delta_m} x^T A y.$$

- Projected Optimistic GDA not clear if works… Let's do Optimistic MWU!

$$\boxed{\begin{aligned} x_i^{t+1} &= x_i^t \frac{1+2\eta(Ay^t)_i - \eta(Ay^{t-1})_i}{\sum_j x_j^t(1+2\eta(Ay^t)_j - \eta(Ay^{t-1})_j)}, \\ y_i^{t+1} &= y_i^t \frac{1-2\eta(A^\top x^t)_i + \eta(A^\top x^{t-1})_i}{\sum_j y_j^t(1-2\eta(A^\top x^t)_j + \eta(A^\top x^{t-1})_j)}. \end{aligned}}$$

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Theorem (Convergence). *Let A be the payoff matrix of a zero sum game and the game has a unique Nash equilibrium. It holds that for η sufficiently small (depends on n, m, A, η can be exponentially small in n, m), starting from uniform distribution* $\lim_{t\to\infty}(x^t, y^t) = (x^*, y^*)$ *under OMWU dynamics*

Min-max in general settings

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Definition (Local Nash). *A critical point* (x^*, y^*) *is a* local Nash *if there exists a neighborhood* U *around* (x^*, y^*) *so that for all* $(x, y) \in U$ *we have that*

$$f(x^*, y) \leq f(x^*, y^*) \leq f(x, y^*).$$

- Does there always exist a local Nash? Is it a good solution concept?

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- Does

No! Not sure…

Min-max in general settings

Theorem (Local Convergence). *Under some mild assumptions on $f(x,y)$ and step-size we have*

$$\textit{Local Nash} \quad \subset \quad \textit{GDA-stable} \ \subset \ \textit{OGDA-stable}$$

Remarks

- This is a local result!
- Unfortunately the inclusions can be strict!

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Remarks

- This is a local result!
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Lemma (Inclusion strict). *There are functions with critical points that are GDA-stable but not local Nash. An example is* $f(x,y) = -\frac{1}{8}x^2 - \frac{1}{2}y^2 + \frac{6}{10}xy$.

Min-max in general settings

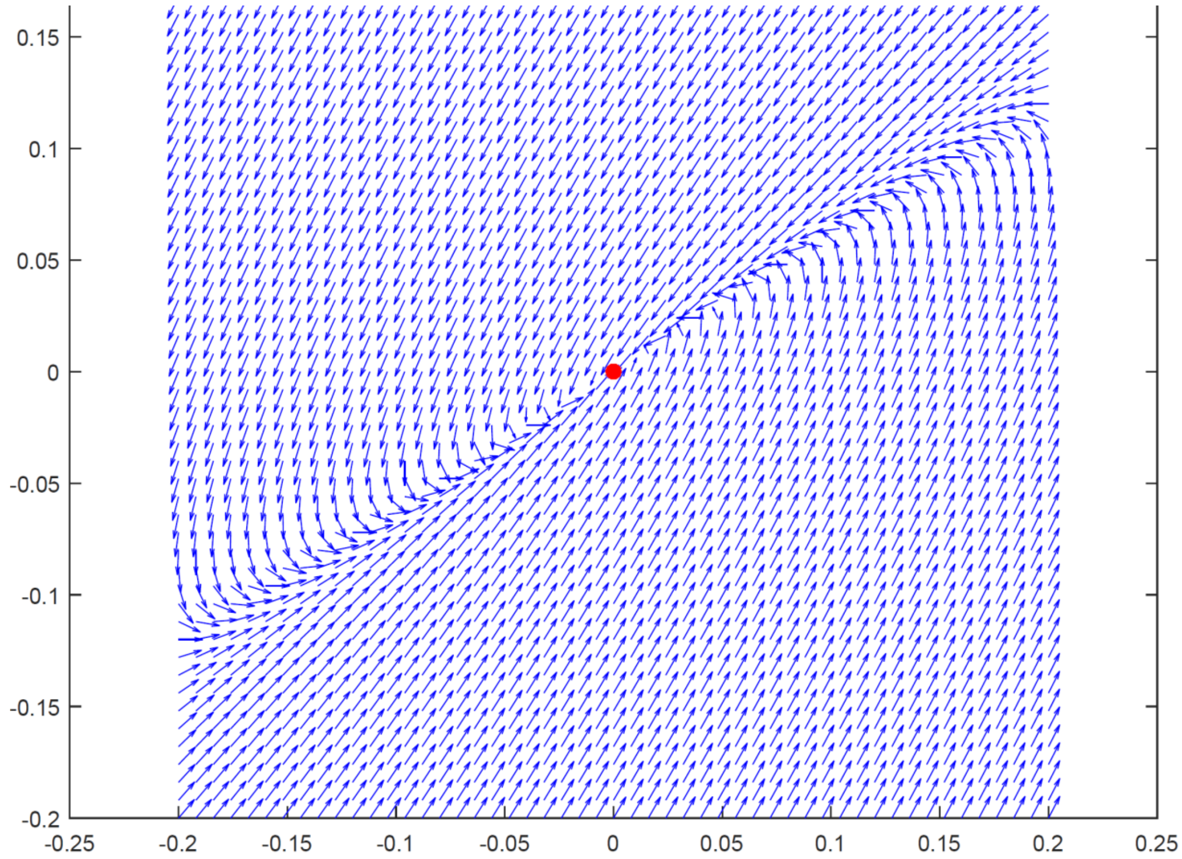
Proof. Let $f(x, y) = -\frac{1}{8}x^2 - \frac{1}{2}y^2 + \frac{6}{10}xy$.
Computing the Jacobian of the update rule of OGDA at $(0, 0)$ we get

$$J_{\mathrm{GDA}} = \begin{pmatrix} 1 + \frac{1}{4}\eta & -\frac{6}{10}\eta \\ \frac{6}{10}\eta & 1 - \eta \end{pmatrix}$$

Both eigenvalues of J_{GDA} have magnitude less than 1 (for any $0 < \alpha < 1.34$). GDA is contracting around $(0, 0)$.

However it is clear that $(0, 0)$ is not a local Nash. Why?

Min-max in general settings



Conclusion

- Introduction to min-max optimization.
  - Negative momentum for last iterate convergence.
  - Bilinear unconstrained and constrained
  - Local Nash

- Next lecture we will talk about Multi-Armed Bandits.